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A note on trees in conjugate Banach spaces

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ABSTRACT

A simple proof is given of the following result due to Stegall: if a Banach space X contains a separable subspace Y with non-separable conjugate Y^*, then for each ε such that $0<\varepsilon<1$, the unit ball $B(X^*)$ of X^* contains an infinite $\varepsilon/2$-tree.

INTRODUCTION

Banach spaces with the Radon-Nikodým property (*RNP*) can be characterized in the following two strikingly different ways: The first one, due to Chatterji [2], states that a Banach space X has the *RNP* if and only if each X-valued uniformly bounded martingale converges almost surely. The second characterization, which is the result of the combined efforts of Rieffel, Maynard, Huff, Davis and Phelps, states that a Banach space X has the *RNP* if and only if each bounded subset of X is dentable. It follows from either of these characterizations that a Banach space with the *RNP* cannot contain a bounded infinite ε-tree for any $\varepsilon>0$. For, such a tree provides, at the same time, a bounded martingale that diverges almost surely and a non-dentable bounded set. The converse is false in general: Bourgain and Rosenthal constructed in $L^1[0,1]$ a closed subspace without the *RNP* in which no bounded infinite ε-tree exists [1]. On the other hand, things are much more pleasant in conjugate Banach spaces. For, Stegall [9] proved that the following conditions are equivalent for an arbitrary Banach space X:

(i) The conjugate space X^* has the *RNP*.
(ii) X^* contains no bounded infinite ε-tree for any $\varepsilon>0$.

(iii) For each separable linear subspace Y of X, its conjugate space Y^* is separable.

We recall that an *infinite tree* in a linear space E is a sequence $\{x_n : n=1,2,\dots\}$ in E such that $x_n = \frac{1}{2}(x_{2n}+x_{2n+1})$ for each n. If E is a normed linear space and $\|x_{2n}-x_n\| = \|x_{2n+1}-x_n\| \geqq \delta$ holds for each n, then $\{x_n\}$ is called an *infinite δ-tree*. We refer to the monograph [4] for all notions unexplained here and a thorough discussion of martingales and dentable sets.

The implication (i)$\Rightarrow$(ii) was already noted. The implication (iii)$\Rightarrow$(i) is a consequence of the Dunford-Pettis theorem and the fact that the *RNP* is separably determined (see e.g. [4]). It is also a fairly straightforward consequence of either one of the characterizations of the *RNP* mentioned above (cf. [5], [3] and [7]). The proof of (ii)$\Rightarrow$(iii), due to Stegall [9] remains considerably more complicated. It is our purpose in this note to give a simple proof of this implication. Our proof also yields another theorem by Stegall [10], namely, a Banach space X is an Asplund space if and only if X^* has the *RNP*, without resorting to Stegall's construction in [9].

We begin with the following simple but useful lemma.

1. LEMMA. *Let $\{K_n : n=1,2,\dots\}$ be a family of non-void compact convex subsets of a linear topological space E such that $K_{2n}\cup K_{2n+1}\subset K_n$ for each n. Then there is an infinite tree $\{x_n\}$ in E such that $x_n\in K_n$ for each n.*

PROOF. Let $Q=\Pi\{K_n : n=1,2,\dots\}$, and for each n let

$$A_n=\{q\in Q : \tfrac{1}{2}(q(2n)+q(2n+1))=q(n)\}.$$

Then A_n is a closed subset of Q, and the conclusion of the lemma is equivalent to: $\cap\{A_n : n=1,2,\dots\}\neq\emptyset$. By compactness it suffices to prove that $A_1\cap\dots\cap A_k\neq\emptyset$ for each k. We fix k and define an element p in Q as follows: For $n>k$, $p(n)$ is chosen arbitrarily in K_n. From $n=k$ to $n=1$, we use the following inductive definition: Suppose that, for $m>n$, $p(m)\in K_m$ is defined; then let $p(n)=\frac{1}{2}(p(2n)+p(2n+1))$. Since K_n is convex and $K_{2n}\cup K_{2n+1}\subset K_n$, $p(n)\in K_n$. Clearly $p\in A_1\cap\dots\cap A_k$, and hence the proof is complete.

2. PROPOSITION. *Let X be a Banach space. If there exists a bounded set B in X^* and an $\varepsilon>0$ such that diam $U>\varepsilon$ whenever U is a non-empty relatively weak*-open subset of B, then w^*-cl co B ($=$weak*-closed convex hull of B) contains an infinite $\varepsilon/2$-tree.*

PROOF. We construct a sequence $\{U_n\}$ of non-empty relatively weak*-open subsets of B and a sequence $\{x_n\}$ in X such that

(a) $\|x_n\|=1$ $(n=1,2,\dots)$

(b) $U_{2n}\cup U_{2n+1}\subset U_n$ $(n=1,2,\dots)$, and

(c) for each n, if $f\in U_{2n}$ and $g\in U_{2n+1}$, then $(f-g)(x_n)\geqq\varepsilon$.

First we let $U_1=B$. Suppose that, for some positive integer m, U_k is defined for $1\leqq k<2^m$ and x_n for $1\leqq n<2^{m-1}$ so that (a), (b) and (c) are valid for all

n such that $1 \leqq n < 2^{m-1}$. Let $2^{m-1} \leqq k < 2^m$. Then by hypothesis diam $U_k > \varepsilon$, and hence there are h_0 and h_1 in U_k such that $\|h_0 - h_1\| > \varepsilon$. Choose $x_k \in X$ so that $\|x_k\| = 1$ and $(h_0 - h_1)(x_k) = \varepsilon + \delta$ for some $\delta > 0$. Let

$$U_{2k} = \{f \in U_k : f(x_k) > h_0(x_k) - \delta/2\}$$

and

$$U_{2k+1} = \{g \in U_k : g(x_k) < h_1(x_k) + \delta/2\}.$$

Then clearly U_{2k} and U_{2k+1} are non-empty relatively weak*-open subsets of B, and U_{2k}, U_{2k+1} and x_k satisfy (a), (b) and (c) for $n = k$. Since $n = 2k$ and $n = 2k+1$ exhaust $\{n : 2^m \leqq n < 2^{m+1}\}$ as k runs through $\{k : 2^{m-1} \leqq k < 2^m\}$, the construction is complete.

Now for each n, let $K_n = w^*\text{-}cl\ co\ U_n$. Then each K_n is non-empty, weak*-compact and convex, and (b) implies that $K_{2n} \cup K_{2n+1} \subset K_n$. Hence by Lemma 1, there is a tree $\{f_n\}$ in X^* such that $f_n \in K_n$ for each n. Since $f_{2n} - f_{2n+1} \in K_{2n} - K_{2n+1} \subset w^*\text{-}cl\ co\ (U_{2n} - U_{2n+1})$, $(f_{2n} - f_{2n+1})(x_n) \geqq \varepsilon$ by (c). Hence $\|f_{2n} - f_{2n+1}\| \geqq \varepsilon$ or equivalently $\|f_{2n} - f_n\| = \|f_{2n+1} - f_n\| \geqq \varepsilon/2$. Hence $\{f_n\}$ is an infinite $\varepsilon/2$-tree in $K_1 = w^*\text{-}cl\ co\ B$.

The following argument, part of which appears in [8], completes our proof of (ii) $\Rightarrow$ (iii).

3. COROLLARY. *Let X be a Banach space, and suppose that X has a separable linear subspace Y with non-separable conjugate Y^*. Then for each ε such that $0 < \varepsilon < 1$, the unit ball $B(X^*)$ of X^* contains an infinite $\varepsilon/2$-tree.*

PROOF. Fix ε such that $0 < \varepsilon < 1$. Then $B(Y^*)$ contains an uncountable set A such that $\|f_1 - f_2\| > \varepsilon$ for any two distinct f_1 and f_2 in A. For, otherwise, $B(Y^*)$ can be covered by countably many balls of radius $\leqq \varepsilon$, each of which can be covered by countably many balls of radius $\leqq \varepsilon^2$, and so on. Since $\varepsilon^n \to 0$ as $n \to \infty$, this would imply that $B(Y^*)$ and hence Y^* are separable. Since $B(Y^*)$ is weak*-metrizable and weak*-separable, all but countably many points of A are weak*-condensation points of A. Deleting at most countably many points from A and taking the weak*-closure, we obtain a weak*-compact subset A_1 of $B(Y^*)$ such that diam $V > \varepsilon$ whenever V is a non-empty relatively weak*-open subset of A_1. Since the restriction map $R : X^* \to Y^*$ is weak*-weak* continuous and $R[B(X^*)] = B(Y^*)$, there is a minimal weak*-compact subset B of $B(X^*)$ such that $R[B] = A_1$. It is easily seen that B satisfies the hypothesis of the proposition. Thus $B(X^*)$ contains an infinite $\varepsilon/2$-tree.

4. REMARKS. We list two further properties equivalent to each of (i), (ii) and (iii) above:

(iv) X is an Asplund space.

(v) Each bounded non-empty subset of X^* contains non-empty relatively weak*-open subsets of arbitrarily small diameters.

It is proved in [8] that (iv) $\Leftrightarrow$ (v). Proposition 2 above states that (ii) $\Rightarrow$ (v), and the argument of Corollary 3 proves (v) $\Rightarrow$ (iii).

Thus we have a relatively easy proof (i.e. independent of Stegall's construction in [9]) of the equivalence of (i)–(v). However, it should be pointed out that our method does not yield a proof of a theorem due to Huff and Morris [6]: X^* has the *RNP* if and only if X^* has the Krein-Milman property. For this, Stegall's construction remains essential.

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